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Monetary Policy under Leviathan Currency Competition

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Abstract

In this paper, we use a dual currency Lagos-Wright model to explore the nature of optimal monetary policy under currency competition using different timing protocols. The central banks are utility maximizing players. To characterize equilibrium with reputation, we model the centralized market sub period of the Lagos-Wright economy as an infinitely repeated game between the two Leviathan central banks (long run players) and a continuum of competitive agents (short run players). Concentrating on Markov strategies in such a game shows that the Markov perfect equilibrium features highest inflation tax. However, allowing for reputation concerns improves the inflation outcome. Such a game typically features multiple equilibriums but the competition between the banks allows the use of renegotiation proof-ness as an equilibrium selection mechanism. Accordingly, equilibrium featuring the lowest inflation tax is weakly renegotiation proof, suggesting that better inflation outcome is more likely in the case of Leviathan currency competition than in the single Leviathan bank case.

JEL Codes: E52, E61.
Keywords: monetary policy, currency competition, Leviathan, inflation tax, money search.

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1. Introduction

In this paper, I extend the model in the Waknis (n.d.) to include a second Leviathan central bank as an additional player making the model a dual currency money search one. I do so to explore if adding a competitive element to the model implies a lower rate of inflation in equilibrium. In countries where central banks are forced to finance government expenditure by printing money, it might be easier to create institutional incentives for lowering wasteful expenditure than disciplining the decision makers explicitly. A model with a Leviathan currency competition is precisely such an experiment in institutional design.

I use different timing protocols to characterize the optimal monetary policy for the competing utility maximizing central banks. As in Waknis (n.d.), the general case of no commitment includes the subcases with and without reputation concerns. We model the centralized market subperiod interaction between the banks and the agents as an infinitely repeated game. Under reputation, using the theory of repeated games for more than one long run player and a continuum of agents, we characterize the equilibrium. Once we know how the agents behave, the one shot game between the banks becomes a prisoners dilemma with a unique Nash equilibrium featuring highest inflation tax. In the infinitely repeated version of the game, there are multiple equilibriums however only the one with lowest inflation tax survives an equilibrium refinement. Thus, unlike the single bank case, the two bank case gives a clearer indication of the surviving equilibrium in infinite repetition.

In the second subcase, the competing banks do not consider reputation while choosing the optimal policy. In such an environment, we use the generalized Euler equation approach to solve for the steady state policy function under Markov perfect equilibrium. Such a policy function suggests that the utility maximizing choice of monetary growth rate is an inverse function of the real value of the competing currency. However any further inference from this policy function requires the use of numerical methods. We then use the repeated game structure to come up with an alternative characterization of the Markov perfect equilibrium. In order to do so we only concentrate on Markov strategies for the central banks. Application of the theory suggests that the Markov perfect equilibrium characterized in this way features highest inflation tax. Thus, it seems allowing reputation concerns is the only way of raising the likelihood of a lower inflation outcome in a dual Leviathan currency competition economy.

The dual currencies approach is not a new one in the money search literature, though. For example, Trejos and Wright (2001) construct a two country, two currency search theoretic model of monetary exchange to analyze the features of the environment that make it more likely that a given money circulates internationally. Head and Shi (2003) use the dual currency setup to determine exchange rate between the two nominal fiat monies. They construct a two country search model for the purpose. Camera et al. (2003) develop a search economy with two currencies, one safe and other risky, and derive conditions for optimal spending preferences and patterns. They also address the behavior of velocity of the two monies under differing difficulty and risk of transacting. Lotz (2004) considers the problem of introducing a new currency when there is already one in use. Curtis and Waller (2000) use a search theoretic model of money to explore the conditions under which two currencies, domestic and foreign, will coexist despite the legal restrictions on the use of foreign currency for internal trade. Craig and Waller (2004) use a money search model to explore the phenomenon of dollarization. In their model, two currencies start with the same risk and then the increase in the risk of one of the currency makes it loose value relative to the other one. This is similar to the dynamics of the currency competition set up in this paper. The only difference is the possibility of a better inflation equilibrium arising endogenously because of currency competition. Whereas in their model the safer foreign currency actually substitutes the riskier domestic currency as a medium of exchange.

The model in this paper also speaks to the literature on privatization of money supply. Especially it relates to the broader debate on desirability of monopoly in currency issue given the strong case against it in other areas of economic analysis. The literature on free banking is a good reference for this debate, with a summary argument in Selgin (2008). A closely related result to this paper is Sun (2007), in which competition between multiple private banks (and their monies) improves welfare as against in the case of prohibition of such competition. The paper addresses the issue of monitoring a bank with undiversifiable risk in a model with limited communication and lack of double coincidence of wants.

The rest of the paper is organized as follows. Section two outlines the dual currency model. Section three outlines a Leviathan central bank's decision dynamics under currency competition. Section four analyzes the policy under full commitment and section five does the same under no commitment. Section five concludes.

2. The Model

The model described here is a variant of Lagos and Rocheteau (2008) with two fiat monies. Goods and money are perfectly divisible. There are two subperiods, a day subperiod where special goods are traded in a decentralized market and a night subperiod where a general good is traded in a centralized Walrasian market. The decentralized market is characterized by trading frictions and hence money gets valued for the liquidity services it provides. The night trading, though centralized, is anonymous and is used by agents to trade in the general good and rebalance their portfolios. The economy is characterized by imperfect memory and record keeping to rule out credit transactions as stressed by Kocherlakota (1998) and Wallace (2001). There are two monetary authorities, B_R and B_B issuing R and B currency respectively. New money is issued by the central banks in the centralized market to consume the general good. The following figure gives the working of the model schematically.

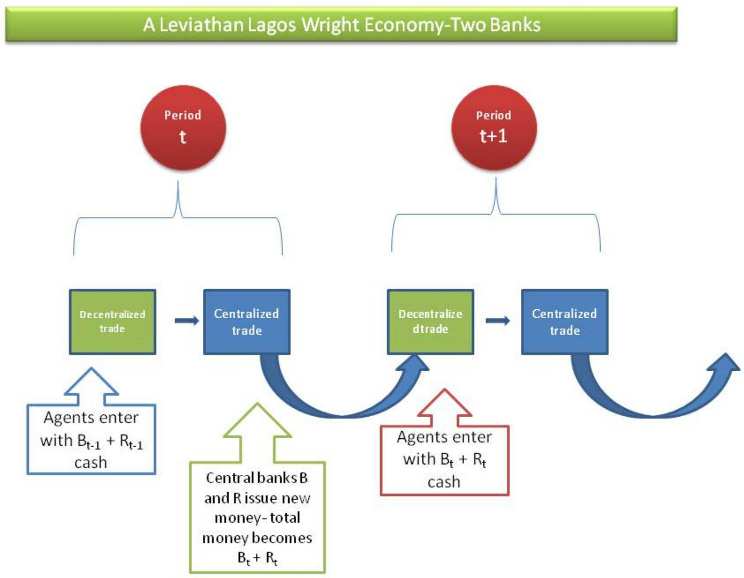

Figure 1: Model Structure

To describe the equilibrium we begin by describing the value functions, taking as given the terms of trade and distribution of monies. The state variables for the individual includes his real money balances and a vector of aggregate states s. Let $s = (\pi^R, \pi^B)$, where π^R and π^B are the growth rates of currency R and B respectively. Let $\phi = (\phi^R, \phi^B)$, where ϕ^R and ϕ^B is the value of money in currency R and B respectively, in the centralized market.

Consider an agent who holds R and B units of currency at the beginning of the first subperiod. Then, his real portfolio is given by $f = [r\ b] = [\phi^R R\ \phi^B B]$. Before entering the decentralized market in the first subperiod the agent decides how much of this money to carry along as a means of payment. He carries both the monies to the market and can pay with either of them or both. Let $a \leq f$ be the portfolio he actually decides to carry into the decentralized market.

In a single coincidence meeting a seller's production h must equal the buyer's consumption x. Let us denote the common value as $q(a, \tilde{a}, s)$ and money that changed hands as $d(a, \tilde{a}, s)$, where a and $\tilde{a}$ are buyer's and seller's real money balances. $B(a, \tilde{a}, s)$ is the payoff in a double coincidence meeting.

Let $V(\mathbf{a}, \mathbf{s})$ be the value function for an agent carrying a portfolio when he enters the decentralized market and $W(a, s)$ be the value function in the afternoon when he enters the centralized market. $V(\mathbf{a}, \mathbf{s})$ is given by,

$$
\begin{aligned}
V(\mathbf{a},\mathbf{s}) &= \alpha\sigma\int\{u[q(\mathbf{a},\tilde{\mathbf{a}},\mathbf{s})]+W[\mathbf{a}-d(\mathbf{a},\tilde{\mathbf{a}},\mathbf{s})]\}dH(\tilde{\mathbf{a}}) \\
&+\alpha\sigma\int\{-c[q(\tilde{\mathbf{a}},\mathbf{a},\mathbf{s})]+W[\mathbf{a}+d(\tilde{\mathbf{a}},\mathbf{a},s)]\}dH(\tilde{a}) \\
&+\alpha\delta\int B(\mathbf{a},\tilde{\mathbf{a}},\mathbf{s})dH(\tilde{\mathbf{a}})+(1-2\alpha\sigma-\alpha\delta)W(\mathbf{a},\mathbf{s})
\end{aligned} \tag{1}
$$

where $H(\cdot)$ is the distribution of currency portfolios. Let $\mathbf{w}$ be the vector of choice variables (r_i, b_i) in the centralized market. The value of entering the centralized market with portfolio $\mathbf{a}$ is

$$
W(\mathbf{a},\mathbf{s}) = \max\{U(X) - AH + \beta V(\mathbf{a_{+1}},\mathbf{s_{+1}})\} \tag{2}
$$

$$
\text{s.t.} X = H + \mathbf{a} - \mathbf{w} \tag{3}
$$

The utility function for the centralized market is quasi linear implying that all the agents will carry the same amount of money out of the centralized market and will also have the same portfolio composition.

Substituting (3) in (2) and normalizing $A = 1$, we get:

$$
W(\mathbf{a},\mathbf{s}) = \mathbf{a} + \max_{X,w}\{U(X) - X - \mathbf{w} + \beta V(\mathbf{a_{+1}},\mathbf{s_{+1}})\} \tag{4}
$$

We can see clearly that $U'(X) = 1$ implying that $X = X^*$. Secondly, the choice of w is independent of a. All that agents care is the real value of their portfolio and not how is constituted. Thirdly, W is linear in a and $W(\mathbf{a}+\mathbf{d},\mathbf{s}) - W(\mathbf{a},\mathbf{s}) = \mathbf{d}$. The payoff in the double coincidence meeting is $B(\mathbf{a},\tilde{\mathbf{a}},\mathbf{s}) = u(q^*) - c(q^*) + W(\mathbf{a},\mathbf{s})$, where no money changes hands.

Using the Nash bargaining solution and following through the algebra (See Appendix 1), we arrive at the following equilibrium conditions for the agent carrying a portfolio of currencies.

$$
\begin{aligned}
z(q_t) - \phi_t^B B_{t-1} &= \beta[z(q_{t+1}) - \phi_{t+1}^B B_t]\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \\
z(q_t) - \phi_t^R R_{t-1} &= \beta[z(q_{t+1}) - \phi_{t+1}^R R_t]\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right]
\end{aligned} \tag{5}
$$

A monetary equilibrium can be characterized as any path for q that stays in $(0, q^*)$ and satisfies (5). The central banks will solve their optimization problem taking the above as a constraint. Thus, we only consider monetary equilibrium in this paper.

3. Central Bank's Problem

As mentioned before, in this model the two central banks are not benevolent. Instead they are utility maximizing. They print money in order to consume in the centralized subperiod of the model. Such banks represent the situation where the government has to resort to seigniorage to finance its consumption. In a version with one Leviathan central bank, Waknis (n.d.) analyzes the optimal monetary policy under different timing protocols. Under most of the protocols, the equilibrium featured a substantial inflation tax. In this paper, we look at what happens to this equilibrium inflation tax in presence of a Leviathan currency competition.

To ensure clear comparison with the single Leviathan central bank case, we consider two cases. In the first case we assume that the central banks can commit to a particular policy and hence choose the policy and therefore sequences of money supply for the life time once for all. In the second case the central banks do not have the ability to commit across periods and hence choose their policy every period. We characterize a Markov perfect equilibrium (MPE) as well as a repeated game equilibria for this case. The MPE concept of equilibrium is a refinement designed to rule out any reputational mechanism that can sustain good equilibria (Klein et al. (2008)). The MPE equilibrium concept itself is due to Maskin and Tirole (2001). The repeated game structure is due to Fudenberg et al. (1990) and Mailath and Samuelson (2006).

4. Full Commitment

Under full commitment, the central bankers choose sequences of money supply for the life time once for all. They does consider (5) while doing so. We are looking for a decision rule of the form $B_{t+1} = g_B B_t$ or $R_{t+1} = g_R R_t$ as the central bank maximizes consumption in the centralized market.

In what follows we will first work out the full commitment policy for $Bank_B$ and the policy for $Bank_R$ works out in a similar way.

Consumption of $Bank_B$ denoted by c_t^B is given as follows:

$$\begin{aligned} c_t^B &= (B_t - B_{t-1})\phi_t^B \\ &= (B_{t-1}(g_B - 1) - B_{t-1})\phi_t^B \\ &= (g_B - 1)B_{t-1}\phi_t^B \\ &= (g_B - 1)g_B^{t-1}B_0\phi_t^B(g_B, g_R) \end{aligned} \tag{6}$$

The central banks take the equilibrium condition of the short lived agents as given while solving it's optimization. This constraint is given by the following:

$$z(q_t) - \phi_t^R R_{t-1} = \beta \frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{(g_B} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \tag{7}$$

Given the above expression for c_t^B, $Bank_B$'s optimization problem can be written as:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \mu^B (c_t^B) && (8) \\ & -\lambda \sum_{t=0}^{\infty} \beta \left\{ \frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{g_B} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})}\right] - (z(q_t) - \phi_t^R R_{t-1}) \right\} && (9) \end{aligned}$$

Solving the above FOC for g_B we get the following (Please refer to the Appendix 3 for working):

$$g_B = \left[\frac{\lambda \left\{ R_t \left[g_B \frac{\partial \phi_{t+1}^R}{\partial g_B} - \phi_{t+1}^R \right] - z(q_{t+1}) \right\}}{\frac{1}{g_B - 1} + \frac{\frac{\partial \phi_t^B}{\partial g_B}}{g_B^{t-1} B_0 \phi_t^B}} (1 + l(q_{t+1})) \right]^{1/2} \tag{10}$$

It can be shown that the above function is a contraction and hence a policy function exists. Having said this, note that g_B is an increasing function of $\phi_{t+1}^R R_t$. The higher the real value of the competing bank's money supply, the lower is the growth rate you can choose. The same will hold for $Bank_R$, with g_R being a decreasing function of $\phi_{t+1}^B B_t$.

5. No Commitment

Under this timing protocol, the central bank cannot commit to any policy. There are two ways in which such a scenario can be analyzed. Even though the central bank does not commit to a monetary policy, it could still care about its reputation. Such a situation could be handled with the help of repeated game theory. This is done in the next subsection. The case where central bank does not consider the effect of its current policies on its reputation(a Markov perfect equilibrium) is analyzed in the following subsection.

5.1. No Commitment- Repeated Game Structure

As argued in Waknis (n.d.), the interaction between the central banks and economic agents in the centralized market can be thought of being repeated infinitely. Hence, we model choice of monetary growth rate under no commitment as an infinitely repeated game. In contrast with the Markov perfect equilibrium, an infinitely repeated game can sustain a reputational equilibrium.

The economic agents continue to behave competitively but the central banks do not. This differential treatment is required to analyze the impact of central banks' choice of their respective monetary growth rate on this economy. We model them as long run player and the economic agents as short run players. The central banks are long run because they anticipate how the future policy depends on the current policy. The short run players take future policies as given.

Fudenberg et al. (1990) provide an analysis of games where the short run players play the game only once and the long run player plays the stage game infinitely. Because the short run players are unconcerned about the future they play their best response moves and hence the equilibrium outcomes lie on their best response functions. Fudenberg and Levine (1994) extend this analysis to the case of more than one player. They consider a general game of imperfect public monitoring with more than one long run player and short run players. We consider a game of perfect monitoring which is a special case of such a game. There have been number of extensions and refinements of such games and Mailath and Samuelson (2006) serves as a point of reference for this literature and the one on repeated games for building reputations. This section uses the modeling apparatus and terminology provided by them.

5.2. Perfect monitoring characterization of games of imperfect public monitoring

In a game with imperfect public monitoring, players $1, ..., n$ are long lived players and players $n+1, ..., N$ are short lived players with player i having a set of pure actions A_i which is a compact subspace of the Euclidian $\Re^k$ for some k. Players choose actions simultaneously.

$\mathbb{B} : \prod_{i=1}^{n} \triangle(A_i) \Rightarrow \prod_{i=n+1}^{N} \triangle(A_i)$ is the mapping of any mixed action profile for the long lived players to the corresponding set of static Nash equilibriums for the short lived players.

At the end of the one shot game players observe a public signal y, drawn from a signal space Y. $\rho(y|a)$ is the probability that the signal y is realized, given the action profile $a \in A = \prod_i A_i$. Player i's payoff after realization (y, a) is given by $u_i^*(y, a)$.

The set of public histories is $\mathbb{H} \equiv \bigcup_{i=0}^{\infty} Y^t$.

The set of histories for player i is $\mathbb{H}_i \equiv \bigcup_{i=0}^{\infty} (A_i \times Y)^t$.

We assume that each short run player in period t only observes the public history.

A pure strategy for player i is a mapping from all possible histories into the set of pure actions, $\sigma_i : \mathbb{H}_i \to A_i$ and a mixed strategy is a mapping $\sigma_i : \mathbb{H}_i \to \triangle(A_i)$.

A behavior strategy σ_i is public if, in every period t, it depends only on the public history $h^t \in Y^t$ and not on i's private history. A perfect public equilibrium (PPE) is a profile of public strategies σ that for any public history h^t, specifies a Nash equilibrium for the repeated game, that is for all t and all $h^t \in Y^t$, $\sigma|_{h^t}$ is a Nash equilibrium. A PPE is strict if each player strictly prefers his equilibrium strategy to every other public strategy. The set of PPE payoff vectors of the long run players is denoted by $\mathcal{E}(\delta) \subset \Re^n$.

Given these definitions, we can describe a game of perfect monitoring as a special case where $Y = A$ and $\rho(y|a) = 1$ if $y = a$ and 0 otherwise. The game between the two central banks (long lived players) and the economic agents (short lived players) in the second subperiod as a game of perfect monitoring.

5.3. Centralized Market Game of Perfect Monitoring

After every decentralized trading period, the centralized market opens with two long run players- the central banks, $Bank_B$ and $Bank_R$, and a $[0, 1]$ continuum of economic agents (short run players) [1]. In the stage game the central banks and the agents choose the monetary growth rate and work hours to maximize consumption of the generic good X. The strategy or decision rule for the players are deduced from their utility and value functions. In order to smooth consumption, the decision rule for the agents asks them to respond by increasing the hours worked one to one to positive changes in monetary growth rate. As in the single Leviathan bank case, the Friedman rule is not optimal here and hence the hold up problem caused by any increase in rate of monetary growth rate would lead to a decrease in q[2]. Any such decrease in the quantity traded in the decentralized market along with the increase in work hours for the agents in the centralized market can be interpreted as an *inflation tax*. The decision rule for the central banks asks them to choose a higher monetary growth rate to increase its consumption.

Each player's payoff depends on his own actions, the action of the central bank and the average of the small players'(all economic agents) actions. All players maximize average discounted sum of payoffs. The economic agents, being small or short lived players, do not affect the distribution of their actions and hence are "anonymous" and optimize myopically.

The agents choose their myopic best responses determining the payoffs for the central banks. If this was not the case then the equilibrium payoff range could be different. Thus, introducing short lived players restricts the possible payoffs for the long-lived player (Mailath and Samuelson, 2006, p.61-67).

The histories include the plays of the long run players (central banks) and only the distri-

[1] I use long run and short run players interchangeably with large and small players and long lived and short lived players. In all the cases the reference is to the time horizon over which optimization takes place.

[2] See Waknis (n.d.) for a numerical example and graphical analysis of inflation tax in an economy with one Leviathan bank. Ideally one should be able to replicate the exercise for two banks, but the issue of indeterminacy of exchange rate remains a major stumbling block. The important point is that the Friedman rule remains infeasible in this economy because the monetary authorities happen to be utility maximizing ones. Hence, we can safely assume a negative effect of decrease in value of the portfolio of currencies on the quantity traded in the decentralized market.

bution of play produced by the small players (agents). For simplicity, we assume perfect monitoring, in the sense that at the end of each period the small economic agents can observe the actions of the central bank and the central bank can observe the aggregate of the economic agents actions.

Consider the following one shot or stage game.

Players: Two central banks (long run players) and $[0, 1]$ continuum of anonymous agents (short run player).

Actions: Central banks choose monetary growth rate, $\pi_i \in (A_i) = [\pi_i^{min}, \pi_i^{max}]$ and the agents choose $H \in [H^{min}, H^{max}]$ to maximize utility in the centralized market. The action set is either finite or a convex subset of the Euclidean subspace $\Re^k$ for some k. $Y = A$ and $\rho(y|a) = 1$ if $y = a$ and 0 otherwise. In addition, μ_i is quasiconcave in π_i.

Payoffs: Agents: $U(H, \pi^R, \pi^B) = U(X^* - H - \pi_t^R \phi_t^R R_{t-1} - \pi_t^B \phi^B B_{t-1})$

Payoffs: Central banks: $\mu^R(R_{t-1}, \pi_t^R, \pi_t^B)$ and $\mu^B(B_{t-1}, \pi_t^R, \pi_t^B)$.

Preferences: Preferences are given by the payoff functions. They are decreasing in H and π for the agents and increasing in own monetary growth rate but decreasing in the other bank's monetary growth rate for the central banks.

Proposition 1. (Equilibrium-Stage game under no commitment): $(H^{max}, \pi_{max}^R, \pi_{max}^B)$ *is the only Nash equilibrium of the stage game.*

Proof of Proposition 1. The agents myopically optimize by playing their best response for the stage game. If we plot the locus of the actions of agents against the locus of actions for the central bank, then till the maximum hours bind they both lie on the 45^o line (Please see Appendix 3 for the figure). Any monetary growth rate off the line combined with hours worked on the line or any hours worked off the line combined with monetary growth rate on the line are not included in the best response function of the agents or the central bank. Thus, the 45^o line also traces all possible equilibriums. However, only one of them is a Nash equilibrium of the stage game.

Note that because the agents always play their best responses, we only have to see if the central banks have any incentive for a unilateral deviation. Given this, the nature of the game between the banks is that of a prisonner'd dilemma. To see this, note that each of bank has a strong incentive to deviate from the minimum required monetary growth rate, assuming that the other bank will stay put. This means that even though there is a better equilibrium, $(H^{max}, \pi_{max}^R, \pi_{max}^B)$ is the only Nash equilibrium of the stage game. **QED**.

What happens if this stage game is repeated infinitely? First and foremost we will have to specify a 'strategy' for the players. Given that the agents are modeled as short run players they myopically optimize and play their Nash responses all the time. That sums up their complete plan of action . Before we comment on the banks' behavior in infinite repetition, we need to deal with some technicalities of having two long run players and a continuum of agents as short run players in the centralized market game.

In the general case with infinitely repeated games, there are multiple equilibriums associated with different degrees of patience. This is the implication of the folk theorem for infinitely repeated games, which, however cannot be applied to the game being considered here. This is because the presence of short lived players means that some of the extreme points of the equilibrium payoff set can only be produced by mixtures and hence do not have corresponding pure strategy representation. In spite of this issue we can still characterize the equilibrium payoff set and derive a version of folk theorem applicable to an infinitely repeated game with long run and short run players. As mentioned earlier, this was done by Fudenberg et al. (1990) for a single long run player and by Fudenberg and Levine (1994) for more than one long run players. In what follows we will use this characterization to determine the equilibrium payoff set for the centralized market game with two central banks.

We will represent the equilibrium set of strictly individually rational payoffs as $\mathcal{F}^{\dagger}$. This set is contained in the set of payoffs that is generated by all the pure actions $\mathcal{F}$. The characterization of these sets for the case of more than one long lived player and a sequence of short lived players uses the concept of a *half space*.

A half space is defined on a metric space comprised of the set of length-l ordered lists of elements of $\Re$ and Euclidian distance. For every nonzero $\mathbf{y} \in \Re^l$ and $r \in \Re$, we have four half-spaces, $H_y \leq (r) = \{\mathbf{x} \in \Re^l : \mathbf{y}.\mathbf{x} \leq r\}$, $H_y < (r) = \{\mathbf{x} \in \Re^l : \mathbf{y}.\mathbf{x} < r\}$, $H_y \geq (r) = \{\mathbf{x} \in \Re^l : \mathbf{y}.\mathbf{x} \geq r\}$ and $H_y > (r) = \{\mathbf{x} \in \Re^l : \mathbf{y}.\mathbf{x} > r\}$ (Corbae et al., 2009, p.174).

Note that the half spaces are convex and the ones defined by weak inequalities above are closed and the ones defined by strict inequality are open. Also, the intersection of closed convex spaces containing them gives a closed convex hull implying that $\mathcal{F}^{\dagger}$ is the convex hull of $\mathcal{F}$ and hence is the intersection of the closed half spaces containing $\mathcal{F}$.

The set of payoffs decomposable on a half space has a simple dependence on δ and hence will be useful to characterize the folk theorem versions. To define decomposability we will have to define enforceability first.

Let $\mathcal{W}$ be the set of states (for example different monetary growth rates could be different states). For any $\mathcal{W} \subset \Re^n$, a mixed action profile $\alpha \in \mathbb{B}$ is enforceable on $\mathcal{W}$ if there exists a mapping $\gamma : Y \to \mathcal{W}$ such that for all $i = 1, ..., n$ and $a_i' \in A_i$ the expected payoff from a_i' is not greater than payoff from α. We say that the function γ enforces α and the payoff can be denoted by $\mathbf{V}_i$, which can be expressed as follows:

$$\begin{aligned} \mathbf{V}_i(\alpha,\gamma) &= (1-\delta)u_i(\alpha) + \delta \sum_{y\in Y} \gamma_i(y)\rho(y|\alpha) \\ &\geq (1-\delta)u_i(a'_i,\alpha_{-1}) + \delta \sum_{y\in Y} \gamma_i(y)\rho(y|a'_i,\alpha_{-1}) \end{aligned} \tag{11}$$

A payoff vector $v \in \Re^n$ is decomposable on $\mathcal{W}$ if there exists a mixed action profile $\alpha \in \mathbf{B}$ enforced by γ on $\mathcal{W}$ such that the $v_i = \mathbf{V}_i$.

The ex ante stage game payoffs are given as $u_i(a) = \sum_{y\in Y} u_i^*(y,a_i)\rho(y|a)$. It is clear from the definition of $\mathbf{V}_i$ above, that it is strictly smaller than this ex ante stage game payoff. Hence, we can only hope to move as close as possible to the ex ante stage game payoff without compromising on the decomposability. In what follows we will outline the technicalities of doing so.

5.3.1. The Maximal Half Space and the Equilibrium Payoff Set

Given a direction $\lambda \in \Re^n$, and constant $k \in \Re$, $H(\lambda,k)$ denotes the half space such that $v \in \Re^n$: $\lambda.v \leq k$. Here v is a payoff vector chosen to be decomposable on the half space. Let $\mathfrak{B}(\mathcal{W};\delta,\alpha)$ be the set of payoffs that can be decomposed by α on $\mathcal{W}$, when the discount factor is δ. Then, for a fixed λ and α, define:

$$k^*(\alpha,\lambda,\delta) = \max_v \lambda.v \tag{12}$$

subject to $v \in \mathfrak{B}(H(\lambda,\lambda.v);\delta,\alpha)$.

For a payoff $v \in \mathfrak{B}$, if we define $x = \frac{\delta_i(y)-v_i}{(1-\delta)}$ and have $x : Y \to \Re^n$ such that:

$$\begin{aligned} v_i &= u_i(\alpha) + E[x_i(y)|\alpha], \forall v_i \\ v_i &\geq u_i(a_i,\alpha_{-i}) + E[x_i(y)|(a_i,\alpha_{-1}), \forall a_i \in A_i, \forall i \end{aligned} \tag{13}$$

Then, if $\lambda.x(y) = 0$ for all y, we say x orthogonally enforces α in the direction λ. Orthogonal enforceability involves a transfer of continuations from some players to other players, λ being the transfer price. It could be thought of as a coordinate direction if $\lambda = \lambda_i e_i$ for some constant $\lambda \neq 0$ and some i. Here e_j denotes the j^{th} standard basis vector. A direction is $ij - pairwise$ if there are two players, $i \neq j$ such that $\lambda = \lambda_i e_i + \lambda_j e_j$. If for all pairwise directions λ^{ij}, α

is orthogonally enforceable in the direction of λ^{ij}, then α is enforceable in all noncoordinate directions (Mailath and Samuelson, 2006, p.276).

It can be shown that $k^*(\alpha,\lambda,\delta)$ is independent of δ and so can be written as $k^*(\alpha,\lambda)$. However, $k^*(\alpha,\lambda) \leq \lambda.u(\alpha)$ with the equality holding only if α is orthogonally enforced in the direction λ. Thus, the equilibrium payoff set can be approximated $H(\lambda, k^*(\alpha,\lambda))$, only if α is chosen appropriately. A way out is to choose an action profile that maximizes $k^*(\alpha,\lambda)$, that is set $k^*(\lambda) \equiv \sup_{\alpha\in\mathbf{B}} k^*(\alpha,\lambda)$ and $H^*(\lambda) \equiv H(\lambda, k^*(\lambda))$. This is referred to as the maximal half-space in the direction λ. It is the largest $H(\lambda,k)$ half space with the property that a boundary point of the half space can be decomposed with respect to that half space (Mailath and Samuelson, 2006, p.293).

The coordinate direction $\lambda = -e_j$ corresponds to minimizing central bank j's payoff because $-e_j.v = -v_j$. Given this, $k^*(-e_j) \leq -\underline{v}_j = -\min_{\alpha\in\mathbf{B}} \max_{a_j} \mu_j(a_j, \alpha_{-j})$, where v_j corresponds to the minmax payoff in the single central bank model. It can be shown that for all δ, $\mathcal{E}(\delta) \subset \bigcap_\lambda H^*(\lambda)$.

To guarantee enforceability of α even when it is not an equilibrium, we need that the signals generated by any action α_i be statistically distinguishable from those of any other mixture α_i'. This is ensured if the profile α has a full rank. It has a full rank for player i if the $|A_i| \times |Y|$ matrix $R_i(\alpha_{-i})$ with elements $[R_i(\alpha_{-i})]_{a_i y} \equiv \rho(y|a_i, \alpha_{-i})$ has full row rank. Here A is finite. If this holds for both the central banks (all the long lived players), then α has individual full rank.

We also need to ensure orthogonal enforceability of α in noncoordinate directions. For this it has to have pairwise full rank. Given that A is finite, we say the profile α has pairwise full rank for players i and j if the $(|A_i| + |A_j|) \times |Y|$ matrix has rank $|A_i| + |A_j| - 1$. Also pairwise full rank implies individual full rank.

In the game under consideration, there are two long lived players and the infinitely lived agents are the short lived players. In such games, if A is finite and there is perfect monitoring ($Y = A$ and $\rho(a'|a) = 1$ if $a' = a$ and 0 otherwise), then all action profiles have pairwise full rank for all pairs of long lived players (central banks). This implies that $\bigcap_{\lambda\in\Lambda^n} H^*(\lambda) = \mathcal{F}^\dagger$.

The coordinate direction $\lambda = e_j$ defines the least favorable payoff ($\bar{v_j}$) for a central bank and as mentioned above $k^*(-e_j)$ gives the minmax payoff. Thus, these two give the stretch of the maximal halfspace and given that the equilibrium payoff $\mathcal{F}^\dagger$ can be approximated by the intersection of the maximal halfspace, $k^*(-e_j)$ and $k^*(e_j)$ form the limit points or bounds of this equilibrium payoff set. With this background and following Proposition 2 from the single central bank case (See Waknis (n.d.)), we can state the following for the two central bank case:

Proposition 2. (Equilibrium Payoff Set): *Minmax and least favorable payoffs for the central bank B are given by* $\underline{v}_B = \pi_B^{min}\phi_B^h B_{t-1}$ *and* $\bar{v}_B = \pi_B^{max}\phi_B^{inln} B_{t-1} - \epsilon$ ***for some*** $\epsilon > 0$***.*** *Similarly, Minmax and least favorable payoffs for the central bank R are given by* $\underline{v}_R = \pi_R^{min}\phi_R^h R_{t-1}$ *and*

$\bar{v}_R = \pi_R^{max}\phi_R^{inln}R_{t-1} - \epsilon$ ***for some*** $\epsilon > 0$.

Note that the presence of ϵ in the equilibrium payoff set represents the fact that the highest consumption is possible only with a mixed strategy and this does not feature within the limit points. This is because, for constructing $\bar{v}$, we are choosing the least favorable mixture for the central bank and then selecting the profile from the short run players' best responses which maximizes this payoff.

Summarizing this discussion, we have the following characterization for more than one long lived player and one or more short lived players:

Theorem 1. *Suppose A is finite and $\mathfrak{F}^\dagger$ has nonempty interior. For every $v \in int\mathfrak{F}^*$ satisfying $v_i < \bar{v}_i$ for $i = 1...,n$, there exists a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium of the repeated game with perfect monitoring with value v.*

Thus. as in the case of the single long lived player, there are multiple equilibriums in the two Leviathan central banks model. Any of these equilibriums can be sustained depending on the level of patience of the central banks. Applying the theorem to the two Leviathan central bank case we get:

Proposition 3. (Long run-Short run player theorem for the Two Leviathans LW Economy): *Suppose there are two Leviathan central banks (long lived players) along with a continuum of agents (short lived players). With the equilibrium payoff set as defined in Proposition 2, for every $v \in int\mathfrak{F}^*$, satisfying $v_i < \bar{v}_i$ for $i = (B, R)$, there exists a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium of the repeated game with perfect monitoring with value v.*

From Figure (2), we can see that much of the dynamics remains the same with two central banks with one major difference. The stage game Nash equilibrium features the minimum inflation tax in case of two central banks case as against the highest inflation tax in the single central bank case. In the two central bank case, it is also interesting to note that the value of a single currency declines faster for any change in the monetary growth rate for a given growth rate of the competing currency (the green curve-two central banks lies lower compared to the red curve-one central bank). The equilibrium payoff set is also scaled down compared to the one in single central bank case, implying the pay off from choosing any given monetary growth rate under currency competition is lower than under a monopoly. This happens because under currency competition demand for any single currency will be lower (assuming equal growth rates) than in the case of currency monopoly. The seigniorage maximizing monetary growth rate is also lower under currency competition.

5.3.2. Equilibrium Selection for the Infinitely Repeated Game

As mentioned above there are multiple equilibriums even in the case of two central banks. Is there any way we can say which one will survive in infinite repetition? Because the stage game is essentially a prisoners dilemma, we know that cooperation (printing money at the lowest rate) can be sustained as an equilibrium. However, that does not mean that it would be 'the equilibrium' that would be played in infinite repetition. A way out is to check if any of the equilibriums could be selected based on some refinement mechanism. In what follows we use this method to show that the better equilibrium $(h^{min}, \pi^R_{min}, \pi^B_{min})$ is the one that is played in infinite repetition.

We use the idea of renegotiation proof equilibrium in the proof. A subgame perfect equilibrium σ is weakly renegotiation proof (WRP) if there do not exist continuation equilibria σ^1, σ^2 of \$sigma$such that \sigma^1$ strictly pareto dominates σ^2. If an equilibrium σ is WRP, then the associated payoffs are also WRP (Farrell and Maskin (1989)).

Proposition 4. (Equilibrium Selection for Infinitely Repeated Game of Two Leviathans LW Economy): *Even though there are multiple equilibriums following Proposition 3, the only equilibrium that will survive in infinite repetition is the equilibrium, $h^{min}, \pi^R_{min}, \pi^B_{min}$, featuring lowest inflation tax.*

Proof of Proposition 4. Farrell and Maskin (1989) and Eric and van Damme (1989) show that

cooperation in a infinitely repeated prisoners dilemma is weakly renogtiation-proof outcome if the discount factor is sufficiently close to 1. Given that the stage game here is a prisoners dilemma, the cooperative equilibrium, $h^{min}, \pi^R_{min}, \pi^B_{min}$, becomes a WRP if the both the central banks are perfectly patient. If this is the case, then $(h^{min}, \pi^R_{min}, \pi^B_{min})$ will be the equilibrium played in the infinite repetition.

QED

5.4. No Commitment-Markov Perfect Equilibrium

The central banks, $Bank_B$ and $Bank_R$, choose a monetary growth rate to maximize utility $\mu(c^b_t)$ and $\mu(c^R_t)$ respectively, from consumption in the centralized market. Let us work with $Bank_b$'s problem first and $Bank_R$'s problem would work out similarly.

The value function for $Bank_B$ can be written as follows:

$$C^B_t(B_{t-1}, R_{t-1}) = \max_{\pi^B_t} \left[\mu(c^B_t) + \beta C^B_{t+1}(B_t, R_t)\right] \tag{14}$$

where (B_{t-1}, R_{t-1}) are the state variables and π^B_t is the control variable.

The consumption of $Bank_b$ can be expressed as follows:

$$\begin{aligned} c_t^B &= (B_t - B_{t-1})\phi_t^B \\ &= (B_{t-1}(1+\pi_t^b) - B_{t-1})\phi_t^B \\ &= \pi_t^B[z(q_t) - \phi_t^R R_{t-1}] \\ &= [G^B(B_{t-1}, R_{t-1}) - 1][z(q_t) - \phi_t^R R_{t-1}] \end{aligned} \tag{15}$$

where $1 + \pi_t^B = G^B(B_{t-1}, R_{t-1})$ is the policy rule that the $Bank_B$ solves for while optimizing. It is assumed that the policy function depends differentiably on the money stock. The agents take this policy rule as given and react accordingly. Considering this, equilibrium q is given by:

$$z(q_t) - \phi_t^R R_{t-1} = \beta \frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{(G^B(B_{t-1}, R_{t-1}))}\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \tag{16}$$

$Bank_B$ solves the optimization problem knowing that (16) holds.

Definition 1. *A Markov-perfect equilibrium is a set of functions $\{C, G\}$: such that for all B and R,*

$$(G^B(B_{t-1}, R_{t-1})) = \arg\max_{\pi^B} \mu(c_t^B) + \beta C(B_t, R_t)$$

subject to

$$z(q_t) - \phi_t^R R_{t-1} = \beta \frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{(G^B(B_{t-1}, R_{t-1}))}\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right]$$

and

$$C_t^B(B_{t-1}, R_{t-1}) = \max_{\pi_t^B}\left[\mu(c_t^B) + \beta C_{t+1}^B(G^B(B_t, R_t))\right]$$

Using this information we can write the central bank's optimization problem as follows:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \mu(c_t^B) \\ &\quad -\beta^{t+1}\lambda_{t+1}\left\{[z(q_t) - \phi_t^R R_{t-1}] - \beta\frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{(G^B(B_{t-1}, R_{t-1}))}\left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right]\right\} \end{aligned} \tag{17}$$

Following Klein et al. (2008) and Martin (2009), we can express the constraint in (17) as $\eta_t(G(M), z(q), \phi) = 0$. Using (Chow, 1997, p.22), we have the following as FOC's for the central bank's problem:

$$\frac{\partial \mathcal{L}}{\partial G^B(B_{t-1})} = \frac{\partial \mu}{\partial G^B(B_{t-1})}\frac{\partial c_t^B}{\partial G^B(B_{t-1})} + \beta\lambda_{t+1}\eta_{t+1}^{G^B(B_{t-1})} = 0 \tag{18}$$

$$\lambda_t = \frac{\partial \mu}{\partial z(q_t)}\frac{\partial c_t^B}{\partial z(q_t)} + \beta\lambda_{t+1}\eta_{t+1}^{z(q_t)} \tag{19}$$

Adding the two FOC's above gives us what is called the **generalized Euler equation (GEE)** for the $Bank_B$ [3]. It captures the tradeoff we are interested in.

$$\frac{\partial \mu}{\partial G^B(B_{t-1})}\frac{\partial c_t^B}{\partial G^B(B_{t-1})} = \left[\lambda_t - \beta\lambda_{t+1}\eta_{t+1}^{G^B(B_{t-1})}\right] - \left[\beta\lambda_{t+1}\eta_{t+1}^{z(q_t)} + \frac{\partial \mu}{\partial z(q_t)}\frac{\partial c_t^B}{\partial z(q_t)}\right] \tag{20}$$

(20) above captures the tradeoff from selecting a higher monetary growth rate today in the following sense. The terms in the first parentheses on the right capture the net effect of choosing a policy of higher consumption today. The first term is the marginal value of the relaxed constraint today and the second term is the discounted marginal value of tightened constraint tomorrow. A different policy that allows for higher consumption today means higher inflation next period and hence a tighter constraint. The increase in utility from choosing such policy today is therefore, the sum of these two effects adjusted for the change in net utility from equilibrium conditions of the agents (second parentheses on the right).

A comparison with GEE for a single bank suggests that marginal utility of increased consumption today in a model with two banks is different than in the one with a single bank. The marginal utility of consumption in this model not only depends on the value of your currency, but also on the value of the competing currency. To see this, using the definition of η, we can write (20) as follows:

$$\frac{\partial \mu}{\partial G^B(B_{t-1})}\frac{\partial c_t^B}{\partial G^B(B_{t-1})} = \left[\lambda_t - \beta\lambda_{t+1}\frac{z(q_{t+1}) - \phi_{t+1}^R R_t}{((G^B(B_{t-1}, R_{t-1})))^2}\beta\left[1 + l(q_{t+1})\right]\right] - \left[-\beta\lambda_{t+1} + \frac{\partial \mu}{\partial z(q_t)}\frac{\partial c_t^B}{\partial z(q_t)}\right] \tag{21}$$

On the other hand, GEE for the single bank in one Leviathan bank model is given by:

$$\frac{\partial \mu}{\partial G(M_{t-1})}\frac{\partial c_t^b}{\partial G(M_{t-1})} = \left[\lambda_t - \beta\lambda_{t+1}\frac{z(q_{t+1})}{((G(M_{t-1})))^2}\beta[1 + l(q_{t+1})]\right] - \left[-\beta\lambda_{t+1} + \frac{\partial \mu}{\partial z(q_t)}\frac{\partial c_t^b}{\partial z(q_t)}\right] \tag{22}$$

In the first parentheses on the right hand side, the value of competing bank's money supply ($\phi_{t+1}^R R_t$) impacts the marginal utility of increased consumption today.

[3] It is called so due to the presence of the derivative of an equilibrium function. (Martin, 2009, p.8)

Following similar procedure we can write down the GEE for $Bank_R$ as follows:

$$\frac{\partial\mu}{\partial G^R(R_{t-1})}\frac{\partial c_t^R}{\partial G^R(R_{t-1})} = \left[\lambda_t - \beta\lambda_{t+1}\eta_{t+1}^{G^R(R_{t-1})}\right] - \left[\beta\lambda_{t+1}\eta_{t+1}^{z(q_t)} + \frac{\partial\mu}{\partial z(q_t)}\frac{\partial c_t^R}{\partial z(q_t)}\right] \quad (23)$$

5.4.1. Steady State Markow Policy

Given that the central banks trades off the current benefit versus the future losses of increasing the monetary growth rate today, how does the optimal policy look like in steady state? To solve for steady state policy function we solve the system of equations comprised of (16) and the first order conditions given by (18) dropping the time subscripts from the relevant variables (Chow, 1997, p.23). We obtain the following expression for the steady state policy function under no commitment (See Appendix A for the working).

$$G^B = \left[\frac{(G^B-1)\left\{[\beta z(q) + R_t(\frac{\partial\phi^R}{\partial G^B} - \phi^R)(1+l(q))] + G_B^2\frac{\partial\phi^R}{\partial G^B}\right\}}{\beta(1+l(q)) - G^B}\right]^{1/3} \quad (24)$$

As expected, the Markow policy for one bank depends on what the other bank does along with other factors like the liquidity premium. The real value of the competing currency impacts the growth rate negatively. Higher the real value of the competing currency ($\phi^R R_t$), lower is the growth rate consistent with the Markow policy. It is clear that the growth rate is function of itself and hence there is a need to solve for the policy function using computational methods. All the factors on the right hand side are a function of the amount of currency in circulation (both blue and Red).Given this, it can be shown that (24) is a contraction and hence the policy function exists.

5.4.2. Markov perfect equilibrium using the repeated game

We can now use the game described in the previous section to further characterize the Markov perfect equilibrium. The stage game between the central banks and the agents changes from period to period, at least for the central banks in terms of the payoffs. Hence, we can think of this game as a dynamic game. Typically in such games the current state and actions determine the current payoffs ruling out reputational dynamics.

We have to show that the decision rules or strategies specified for both these players are Markov strategies. The decision rule for the central banks is to increase the monetary growth rate to increase the consumption. The decision rule for the agents says that they have to work more if the monetary growth rate increases.

The strategy profile σ is a Markov strategy if for any two ex post histories, $\tilde{h}^t$ and $\tilde{h}^\tau$ of the same length and terminating in the same state, $\sigma(\tilde{h}^t) = \sigma(\tilde{h}^\tau)$. The strategy profile σ is a stationary Markov strategy if for any two ex post histories, $\tilde{h}^t$ and $\tilde{h}^\tau$ (of equal or different lengths) and terminating in the same state, $\sigma(\tilde{h}^t) = \sigma(\tilde{h}^\tau)$.

Proposition 5. (Markov and Stationary Markov Strategies): *The decision rule for the central banks in the stage game above is Markov as well as stationary Markov.*

Proof of Proposition 5. Let there be two expost histories, $\tilde{h}^1$ and $\tilde{h}^2$ of either similar or different length. Let σ = (increase the money growth rate to increase consumption) and σ' = (do nothing). Also, let the states be ordered according to the consumption from lower to higher. Suppose that at the start of history $\tilde{h}^1$, consumption is c_1 and at the start of the history $\tilde{h}^2$, consumption is c_2 ending in a state with consumption c_3. Note that because states are ordered, $c_1 < c_2 < c_3$.

It is easy to see that only if σ was followed in both the cases, the state c_3 could be reached. This is assuming that agents still have some room to adjust their work effort (inflation tax). If σ' was followed instead, then the state at the end of histories would be the respective states. This holds for any two histories irrespective of their lengths being equal or not. Hence, σ is not only Markov but stationary Markov. **QED**

What would be the equilibrium of the repeated game if we only consider Markov strategies? In a repeated game with only Markov strategies, a Markov equilibrium must play a stage game Nash equilibrium and for a stationary Markov strategies, it should play the same one in every period (Mailath and Samuelson, 2006, p.178). Thus, given that the unique Nash equilibrium of the stage game in the previous section features the survival monetary growth rate, a stationary Markow perfect equilibrium will always feature the lowest inflation tax. Thus, $(H^{max}, \pi_R^{max}, \pi_B^{max})$ is a Markov perfect equilibrium of the dynamic repeated game.

5.5. Results at a glance

The following table summarizes the results at glance for the all the cases under no commitment for one as well as two central banks.

No. of Leviathan Banks	Stage Game Nash Equilibrium	Markov Perfect Equilibrium	Reputation(LR-SR) Equilibrium
One	High inflation tax	High inflation tax	multiple
Two	High inflation tax	High inflation Tax	Low inflation tax

Under different timing protocols, we characterized the optimal policy for two competing utility maximizing central banks. We demonstrated that he policy under full commitment suffers from time inconsistency and hence is not credible. Under no commitment we considered two cases: in one case reputation concerns were important and in the other they were not. To characterize these cases, we used a game theoretic approach. Under reputation, we modeled the two central banks as long lived players and the agents as a continuum of short lived players and use theorem by Fudenberg and Levine (1994) to show there are multiple equilibriums possible. After showing that the stage game is actually a prisoners dilemma and hence the centralized market game an infinitely repeated one, we used characterization in Farrell and Maskin (1989) to show that the equilibrium with minimum inflation tax is the only one that is weakly renegotiation proof. Hence, this equilibrium is the most likely one to survive infinite repetition.

The no commitment case without reputation was analyzed using the concept of a Markov perfect equilibrium. The generalized Euler equation approach helped us to show that a policy function under Markov perfect equilibrium exists, but any further specification of such a function required use of numerical methods. However, we were able to come up with an alternative analytical characterization of a Markov equilibrium by modifying the game between central banks and the agents. We showed that if all the players were only allowed period by period decisions, then a Markov perfect equilibrium featured highest possible consumption for the central bank and hence highest inflation tax on agents.

6. Conclusion

In this paper, we use a dual currency Lagos-Wright model to explore the nature of optimal monetary policy under Leviathan currency competition. We do so for various timing protocols. Under full commitment, the optimal monetary growth rate is a decreasing function of the real value of the competing currency. Thinking of the centralized market game as a dynamic game and concentrating on stationary Markov strategies allows us to characterize the Markov perfect equilibrium. It features highest inflation tax. If we allow the banks to have reputational concerns, then the Nash equilibrium of the stage game between the two Leviathan central banks and the continuum of competitive agents features highest inflation tax similar to the single Leviathan bank case.

There are multiple equilibriums in the infinite repetition of this game but that does not pose a problem to narrow down the possibilities. The competition between banks and the fact that agents only play Nash responses transforms the centralized market game to a prisoners dilemma and allows us to use the idea of renegotiation proof equilibria as a refinement mechanism. It helps us predict the equilibrium that would survive the infinite repetition of the centralized market game of Leviathan currency competition. The refinement suggests that if the both the banks are patient enough, then the equilibrium with lowest inflation tax is weakly renegotiation proof implying that Leviathan currency competition is more likely to lead to better inflationary outcome than the single bank case.

The model and analysis in this paper serves as an experiment in institutional design to alter the incentives for governments relying on seigniorage to finance their expenditures. In the model with one Leviathan central bank we alluded to an equilibrium featuring lower inflation tax under the threat of competition. This was said to be possible for the case of a self interested Sovereign as well as a private money supplier facing competition. The results in this paper not only confirm that intuition but also reflects positively on the literature pertaining to free banking or competitive money supply.

A

A1. Agent's equilibrium conditions

Now consider a single coincidence meeting. The terms of trade (q, d) are given by generalized Nash bargaining,

$$\max_{q,d\leq a}[u(q)+W(\mathbf{a}-\mathbf{d},\mathbf{s})-W(\mathbf{a},\mathbf{s})]^{\theta}[-c(q)+W(\tilde{\mathbf{a}}+\mathbf{d},\mathbf{s})-W(\tilde{\mathbf{a}},\mathbf{s})]^{1-\theta} \tag{25}$$

where $\theta \in (0,1)$ is the bargaining power of the buyer. Using the fact that $W(a+d,s)-W(a,s)=\mathbf{d}$, Equation (25) can be rewritten as:

$$\max_{q,d\leq a}[u(q)-\mathbf{d})]^{\theta}[-c(q)+\mathbf{d}]^{1-\theta} \tag{26}$$

Note that (26) does not depend on $\tilde{a}$. From the FOC for this problem the value of $q \in (0, q^*)$ that solves the problem is given by:

$$z(q)=\frac{\theta c(q)u'(q)+(1-\theta)u(q)c'q)}{\theta u'(q)+(1-\theta c'(q)}=\mathbf{d} \tag{27}$$

If $\mathbf{a}' \geq z(q^*)$ the buyer receives $q=q^*$. If $\mathbf{a}' < z(q^*)$ then buyer gets the q that solves **z(q)=a′** in exchange for all the money, i.e. **d=a**. It is clear that q only depends on $\mathbf{a}'$ and not on the seller's portfolio or composition of buyer's portfolio.

So the bargaining solution can be written as:

$$q(\mathbf{a}')=\begin{cases} q^* & \text{if } \mathbf{a}\geq z(q^*) \\ z^{-1}(\mathbf{a}) & \text{if } \mathbf{a}< z(q^*) \end{cases} \tag{28}$$

$$\mathbf{d}'=\begin{cases} z(q^*) & \text{if } \mathbf{a}\geq z(q^*) \\ \mathbf{a} & \text{if } \mathbf{a}'< z(q^*) \end{cases} \tag{29}$$

Now we have to find the optimal choice for $\mathbf{a_{+1}}$. Using the linearity of W and the bargaining solution in (28) and (29) above, let us rewrite the $V(\mathbf{a},\mathbf{s})$ as follows:

$$\begin{aligned} V(\mathbf{a},\mathbf{s}) &= \alpha\sigma\{u[q(\mathbf{a})]+\mathbf{d}(a,\tilde{a},s)\} \\ &+\alpha\sigma\int\{-c[q(\tilde{\mathbf{a}},\mathbf{a},\mathbf{s})]+\mathbf{d}(\tilde{\mathbf{a}},\mathbf{a},\mathbf{s}))]\}dH(\tilde{a}) \\ &+\alpha\delta[u(q^*)-c(q^*)]+\mathbf{a}+\{U(X^*)-X^*-\mathbf{w}+\beta V(\mathbf{a_{+1}},\mathbf{s_{+1}})\} \end{aligned} \tag{30}$$

Now let $v(a, s)$ be given by:

$$\begin{aligned} v(\mathbf{a}, \mathbf{s}) \quad &= \quad \alpha\sigma\{u[q(\mathbf{a})] + \mathbf{d}(a, \tilde{a}, s)\} \\ &\quad + \alpha\sigma \int \{-c[q(\tilde{a}, a, s)] + \mathbf{d}'(\tilde{a}, a, s)]\} dH(\tilde{a}) \\ &\quad + \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^* \end{aligned} \tag{31}$$

Then, $V(\mathbf{a}, \mathbf{s})$ can be written as

$$V(\mathbf{a}, \mathbf{s}) = \max_{a_{+1}} \{v(a, s) + \mathbf{a} - \mathbf{w} + \beta V(\mathbf{a_{+1}}, \mathbf{s_{+1}})\} \tag{32}$$

Assuming all the conditions essential for the solution to exist and be unique are met, we have the following for $\mathbf{pa} < z(q^*)$:

$$V_a = \begin{cases} \phi_t^R + \alpha\sigma[u'(q)q'(\mathbf{a}') - \phi_t^R] \\ \phi_t^B + \alpha\sigma[u'(q)q'(\mathbf{a}') - \phi_t^B] \end{cases} \tag{33}$$

Solving further and using (27) we have:

$$V_a = \begin{cases} \phi_t^R \left[(1 - \alpha\sigma) + \alpha\sigma \frac{u'(q)}{z'(q)}\right] \\ \phi_t^B \left[(1 - \alpha\sigma) + \alpha\sigma \frac{u'(q)}{z'(q)}\right] \end{cases} \tag{34}$$

Now consider the problem of the agent who wants to decide on the portfolio to be carried out of the centralized market, $\mathbf{a_{+1}}$. From (32) above, he solves the following:

$$\max_{a_{+1}} \{-\mathbf{w} + \beta V(\mathbf{a_{+1}}, \mathbf{s_{+1}})\} \tag{35}$$

It can be shown that $\phi \geq \beta\phi_{+1}$ and hence $\{-\mathbf{w} + \beta V(\mathbf{a_{+1}}, \mathbf{s_{+1}})\}$ is non increasing in $\mathbf{a_{+1}} > \mathbf{a}^*_{+1}$. Given this it can also be shown that any solution $\mathbf{a_{+1}}$ must be strictly less that $\mathbf{a}^*_{+1}$. This implies $\mathbf{d} = \mathbf{a}$ and $q < q^*$. Moreover, becasue V is strictly concave for $\mathbf{a_{+1}} < \mathbf{a}^*_{+1}$, there exists a unique maximizer $\mathbf{a_{+1}}$. Thus, the distribution of currencies is degenerate at the end of the centralized market.

The first order conditions for the maximization problem above are:

$$FOC = \begin{cases} \phi_t^R + \beta V_r(\mathbf{a_{+1}}, \mathbf{s_{+1}}) \leq 0 \\ \phi_t^B + \beta V_b(\mathbf{a_{+1}}, \mathbf{s_{+1}}) \leq 0 \end{cases} \tag{36}$$

Updating (34) one period ahead and using it in (36), we get:

$$\phi_t^R = \beta\phi_{t+1}^R \left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \tag{37}$$

$$\phi_t^B = \beta\phi_{t+1}^B \left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \tag{38}$$

Using the fact that $\phi_t^R R_{t-1} + \phi_t^B B_{t-1} = z(q_t)$, we get the following difference equations in q for the agent carrying a portfolio of currencies.

$$\begin{aligned} z(q_t) - \phi_t^B B_{t-1} &= \beta[z(q_{t+1}) - \phi_{t+1}^B B_t] \left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \\ z(q_t) - \phi_t^R R_{t-1} &= \beta[z(q_{t+1}) - \phi_{t+1}^R R_t] \left[1 - \alpha\sigma + \alpha\sigma\frac{u'(q_{t+1})}{z'(q_{t+1})}\right] \end{aligned} \tag{39}$$

A monetary equilibrium can be characterized as any path for q that stays in $(0, q^*)$ and satisfies (39).

A2. No Commitment Game of Perfect Monitoring with Two Central Banks

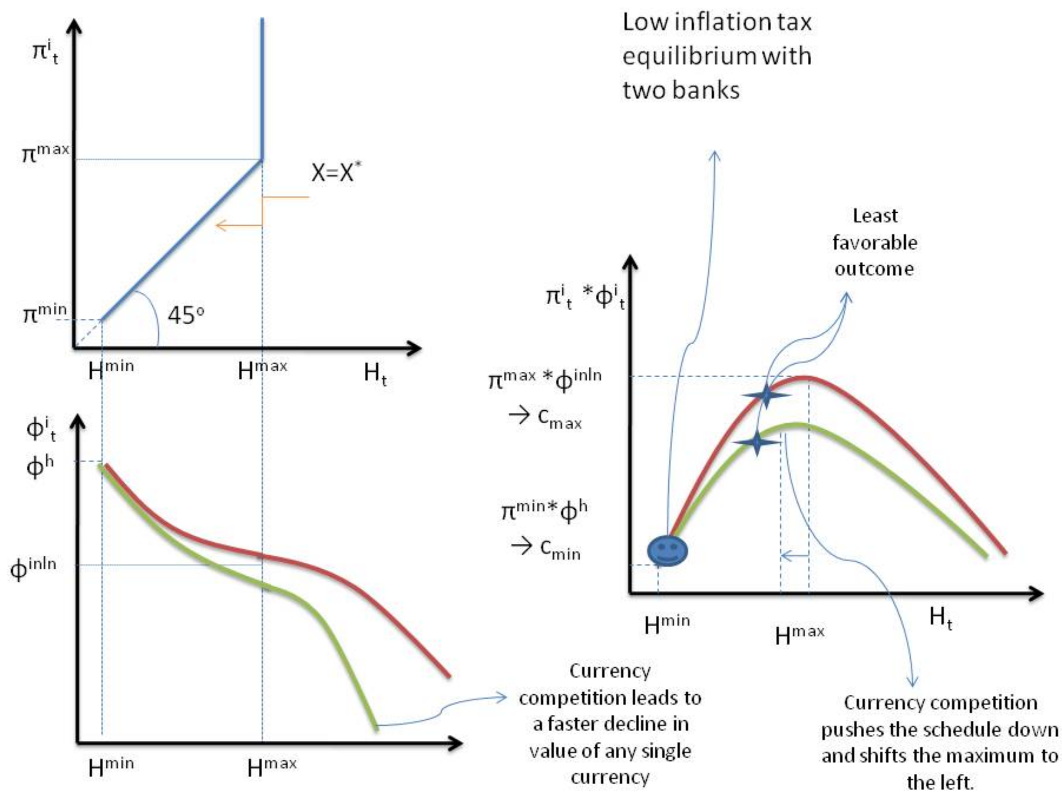

Figure 2: Equilibrium under no commitment: One vs. two Leviathan Central Banks

A3. Full Commitment Policy Function

The first order condition is given by:

$$\frac{\partial \mathcal{L}}{\partial g_B} = \frac{\partial \mu^B}{\partial g_B}\frac{\partial c_B}{\partial g_B} - \lambda \left[\frac{R_t \left[g_b \frac{\partial \phi^R_{t+1}}{\partial g_B} - \phi^R_{t+1} \right] - z(q_{t+1})}{g_B^2} (1 + l(q_{t+1})) \right]$$

Now assuming $\mu^B(c_t^B) = ln(c_t^B)$, we have the following:

$$\frac{\partial \mu^B}{\partial g_B}\frac{\partial c_B}{\partial g_B} = \frac{1}{g_B - 1} + \frac{\frac{\partial \phi^B_t}{\partial g_B}}{g_B^{t-1} B_0 \phi^B_t}$$

Setting the first order condition to zero and substituting second equation in the first we get:

$$g_B = \left[\frac{\lambda \left\{ R_t \left[g_B \frac{\partial \phi^R_{t+1}}{\partial g_B} - \phi^R_{t+1} \right] - z(q_{t+1}) \right\}}{\frac{1}{g_B - 1} + \frac{\frac{\partial \phi^B_t}{\partial g_B}}{g_B^{t-1} B_0 \phi^B_t}} (1 + l(q_{t+1})) \right]^{1/2}$$

Similarly, for $Bank_R$, we get:

$$g_R = \left[\frac{\lambda \left\{ B_t \left[g_R \frac{\partial \phi^B_{t+1}}{\partial g_R} - \phi^B_{t+1} \right] - z(q_{t+1}) \right\}}{\frac{1}{g_R - 1} + \frac{\frac{\partial \phi^R_t}{\partial g_R}}{g_R^{t-1} R_0 \phi^R_t}} (1 + l(q_{t+1})) \right]^{1/2}$$

A4. No Commitment: Steady State Markow Policy

The steady state FOC are as follows:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial G^B} &= \frac{\partial \mu}{\partial G^B}\frac{\partial c^B}{\partial G^B} + \beta\lambda\eta^{G^B} = 0 \\ \lambda &= \frac{\partial \mu}{\partial z(q)}\frac{\partial c^B}{\partial z(q)} + \beta\lambda\eta^{z(q)}\end{aligned}$$

Combining these together and after some manipulation this is equal to:

$$\frac{\frac{\partial \mu}{\partial z(q)}\frac{\partial c^B}{\partial z(q)}}{\frac{\partial \mu}{\partial G^B}\frac{\partial c^B}{\partial G^B}} = \frac{\eta^{z(q)}}{\eta^{G^B}} - \frac{1}{\beta\eta^{G^B}} \tag{40}$$

From the definition of c^B we know that $\frac{\partial c^b}{\partial G^B} = [z(q) - \phi^R R_{t-1}]$ and $\frac{\partial c^B}{\partial z(q)} = [G^B - 1]$. From the definition of η we have the following for the partial derivatives:

$$\begin{aligned}\eta^{z(q)} &= \frac{\beta(1 + l(q))}{G^B} - 1 \\ \eta^{G^B} &= \frac{[\beta z(q) + R_t(\frac{\partial \phi^R}{\partial G^B} - \phi^R)](1 + l(q))}{G_B^2} + \frac{\partial \phi^R}{\partial G^B}\end{aligned}$$

Using the above two equations and after some manipulations we get the following for the steady state Markow perfect policy function:

$$G^B = \left[\frac{(G^B - 1)\left\{[\beta z(q) + R_t(\frac{\partial \phi^R}{\partial G^B} - \phi^R)(1 + l(q))] + G_B^2\frac{\partial \phi^R}{\partial G^B}\right\}}{\beta(1 + l(q)) - G^B}\right]^{1/3}$$

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